FP2 S14 IAL

1. (a) Show that

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)}$$

(2)

(b) Hence, or otherwise, find

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)}$$

giving your answer as a single fraction in its simplest form.

(4)

a) $\frac{1}{2(r+1)(r+2)} - \frac{1}{2(r+2)(r+3)} = \frac{(r+3)-(r+1)}{2(r+1)(r+2)(r+3)}$

$= \frac{2}{2(r+1)(r+2)(r+3)} = \frac{1}{(r+1)(r+2)(r+3)}$ #

b) $\sum_{1}^{n} = \left(\frac{1}{2(2)(3)} - \cancel{\frac{1}{2(3)(4)}}\right) + \left(\cancel{\frac{1}{2(3)(4)}} - \cancel{\frac{1}{2(4)(5)}}\right) + \dots \left(\cancel{\frac{1}{2(n+1)(n+2)}} - \frac{1}{2(n+2)(n+3)}\right)$

$= \frac{1}{12} - \frac{1}{2(n+2)(n+3)} = \frac{(n+2)(n+3) - 6}{12(n+2)(n+3)} = \frac{n^2+5n+6-6}{12(n+2)(n+3)}$

$= \frac{n(n+5)}{12(n+2)(n+3)}$

2. Use algebra to find the set of values of x for which

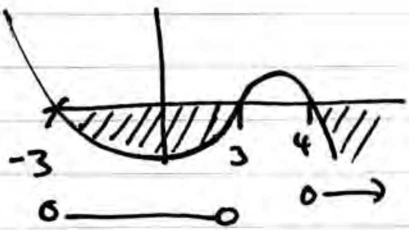
$$\frac{6}{x-3} \leqslant x+2$$

(7)

$$\frac{6}{x-3} - x+2\left(\frac{x-3}{x-3}\right) \leq 0 \Rightarrow \frac{6-(x+2)(x-3)}{(x-3)} \leq 0$$

$$\frac{6-x^2+x+6}{(x-3)} = \frac{-(x^2-x-12)}{(x-3)} = \frac{-(x+3)(x-4)}{(x-3)} \leq 0$$

3, −3, 4



$-3 < x < 3$ $\quad$ $x > 4$

3. Solve the equation

$$z^5 = 16 - 16\mathrm{i}\sqrt{3}$$

giving your answers in the form $re^{\mathrm{i}\theta}$ where θ is in terms of π and $0 \leqslant \theta < 2\pi$.

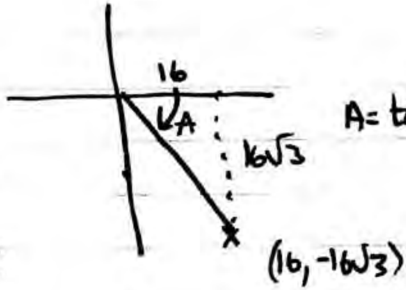
(5)

$$\therefore \quad z^5 = 32\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$$

$$z^5 = 32\left(\cos\left(\frac{-\pi}{3} + 2k\pi\right) + i\sin\left(\frac{-\pi}{3} + 2k\pi\right)\right)$$

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{6k-1}{3}\right)\pi + i\sin\left(\frac{6k-1}{3}\right)\pi\right)^{\frac{1}{5}}$$

$$z = 2\left(\cos\left(\frac{6k-1}{15}\right)\pi + i\sin\left(\frac{6k-1}{15}\right)\pi\right)$$



$A = \tan^{-1}\frac{16\sqrt{3}}{16}$

$A = \frac{\pi}{3}$

$\arg(z) = -\frac{\pi}{3}$ $\quad r^2 = 16^2 + (16\sqrt{3})^2$

$r = 32$

$k=1 \quad z = 2e^{i\frac{5}{15}\pi}$,

$k=2 \quad z = 2e^{i\frac{11}{15}\pi}$

$k=3 \quad z = 2e^{i\frac{17}{15}\pi}$

$k=4 \quad z = 2e^{i\frac{23}{15}\pi}$

$k=5 \quad z = 2e^{i\frac{29}{15}\pi}$

4. A transformation from the z-plane to the w-plane is given by

$$w = \frac{z}{z+3}, \qquad z \neq -3$$

Under this transformation, the circle $|z| = 2$ in the z-plane is mapped onto a circle C in the w-plane.

Determine the centre and the radius of the circle C.

(7)

$wz + 3w = z \Rightarrow z - wz = 3w \Rightarrow z(1-w) = 3w$

$\Rightarrow z = \frac{3w}{1-w} \Rightarrow |z| = \left|\frac{3w}{1-w}\right| \Rightarrow |3w| = 2|1-w|$

$\Rightarrow 3|w| = 2|w-1|$

$\Rightarrow 3|u+iv| = 2|(u-1)+iv|$

$\Rightarrow 3^2(u^2+v^2) = 2^2((u-1)^2+v^2)$

$\Rightarrow 9u^2 + 9v^2 = 4u^2 - 8u + 4 + 4v^2$

$\Rightarrow 5u^2 + 8u + 5v^2 = 4$

$\Rightarrow u^2 + \frac{8}{5}u + v^2 = \frac{4}{5}$

$(u - \frac{4}{5})^2 + v^2 = \frac{4}{5} + \frac{16}{25} = \frac{36}{25}$

$(u - \frac{4}{5})^2 + v^2 = \frac{36}{25}$ Circle $C(\frac{4}{5}, 0)$ $r = \frac{6}{5}$

5.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

(a) Show that

$$\frac{d^4y}{dx^4} = \left(ax^2 + b\right)\frac{d^2y}{dx^2}$$

where a and b are constants to be found.

(5)

Given that $y = 1$ and $\frac{dy}{dx} = 3$ at $x = 0$

(b) find a series solution for y in ascending powers of x up to and including the term in x^4

(5)

(c) use your series to estimate the value of y at $x = -0.2$, giving your answer to four decimal places.

(2)

$$\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - 2\cancel{\frac{dy}{dx}} + 2\cancel{\frac{dy}{dx}} = 0 \qquad \frac{d^3y}{dx^3} = 2x\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^4y}{dx^4} = 2\frac{d^2y}{dx^2} + 2x\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} + 2x\left(2x\frac{d^2y}{dx^2}\right)$$

$$\therefore \frac{d^4y}{dx^4} = \left(4x^2+2\right)\frac{d^2y}{dx^2}$$

$x_0 = 0 \quad y_0 = 1 \quad y_0' = 3 \qquad y_0'' = -2(1) = -2$

$y_0''' = 0$

$y_0'''' = 2(-2) = -4$

$$\therefore y = 1 + 3x - x^2 - \frac{1}{6}x^4$$

c) $x = -0.2 \quad y \simeq 0.3597$

6.

$$x\frac{dy}{dx} + (1-2x)y = x, \qquad x > 0$$

Find the general solution of the differential equation, giving your answer in the form $y = f(x)$.

(9)

$\frac{dy}{dx} + \left(\frac{1-2x}{x}\right)y = 1 \quad$ IF $f(x) = e^{\int \frac{1}{x} - 2 \, dx}$

$= e^{\ln x - 2x} = xe^{-2x}$

$xe^{-2x}\frac{dy}{dx} + \left(\frac{1-2x}{\cancel{x}}\right)\cancel{x}e^{-2x}y = xe^{-2x}$

$\therefore \frac{d}{dx}(xye^{-2x}) = \cancel{x}e^{-2x} \quad \therefore xye^{-2x} = \int xe^{-2x}\,dx$

$u = x \quad v = -\frac{1}{2}e^{-2x}$

$u' = 1 \quad v' = e^{-2x}$

$\Rightarrow xye^{-2x} = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$\therefore xye^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$

$\therefore y = -\frac{1}{2} - \frac{1}{4x} + \frac{c}{x}e^{2x}$

7. The point P represents a complex number z on an Argand diagram, where

$$|z+1| = |2z-1|$$

and the point Q represents a complex number w on the Argand diagram, where

$$|w| = |w-1+i|$$

Find the exact coordinates of the points where the locus of P intersects the locus of Q.

(7)

$$|(x+1)+iy| = |(2x-1)+2iy|$$

$$\Rightarrow (x+1)^2+y^2 = (2x-1)^2+(2y)^2$$

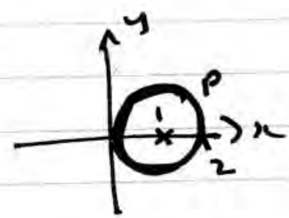
$$\Rightarrow x^2+2x+\cancel{1}+y^2 = 4x^2-4x+\cancel{1}+4y^2$$

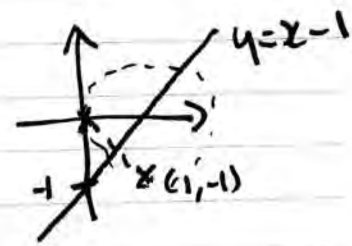
$$0 = 3x^2-6x+3y^2$$

$$\therefore x^2-2x+y^2=0$$

$$(x-1)^2+y^2=1$$

circle $c(1,0)$ $r=1$





$$\therefore (x-1)^2+(x-1)^2=1$$

$$2(x-1)^2=1$$

$$(x-1)^2=\frac{1}{2}$$

$$x=1\pm\frac{1}{\sqrt{2}} \quad y=\pm\frac{1}{\sqrt{2}}$$

$$\left(1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\left(1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

8. (a) Show that the substitution $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \qquad x > 0 \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$$

(7)

(b) Hence find the general solution of the differential equation (I).

(5)

$x = e^t \quad x = e^t \quad t = \ln x$

$\frac{dx}{dt} = e^t \quad x^2 = e^{2t}$

$\frac{dt}{dx} = e^{-t}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \left(e^{-t}\frac{dt}{dx}\right)\frac{dy}{dt} + e^{-t}\frac{dt}{dx}\frac{d^2y}{dt^2} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

$$= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\therefore x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0$$

$$\Rightarrow \cancel{e^{2t}}\left(\cancel{e^{-2t}}\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 5\cancel{e^t}\left(\cancel{e^{-t}}\frac{dy}{dt}\right) + 13y = 0$$

$$\therefore \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0 \quad \#$$

b) cf $y = Ae^{mt} \therefore Ae^{mt}(m^2 + 4m + 13) = 0 \Rightarrow (m+2)^2 = -9 \therefore m = -2 \pm 3i$

$y_{cf} = e^{-2t}(A\cos 3t + B\sin 3t)$

$y = e^{-2(\ln x)}\left(A\cos(3\ln x) + B\sin(3\ln x)\right)$

$$y = \frac{A\cos(3\ln x) + B\sin(3\ln x)}{x^2}$$

9.

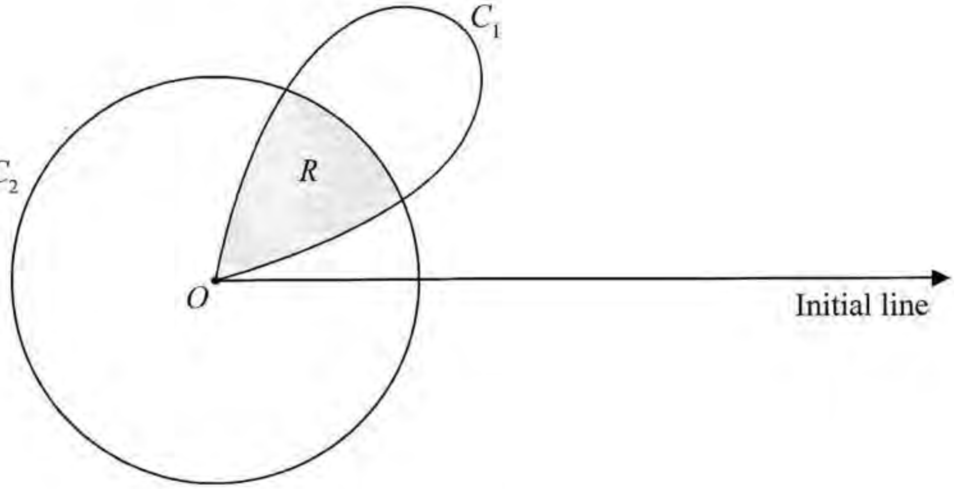


Figure 1

Figure 1 shows the curve C_1 with polar equation $r = 2a\sin 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$, and the circle C_2 with polar equation $r = a$, $0 \leqslant \theta \leqslant 2\pi$, where a is a positive constant.

(a) Find, in terms of a, the polar coordinates of the points where the curve C_1 meets the circle C_2

(3)

The regions enclosed by the curve C_1 and the circle C_2 overlap and the common region R is shaded in Figure 1.

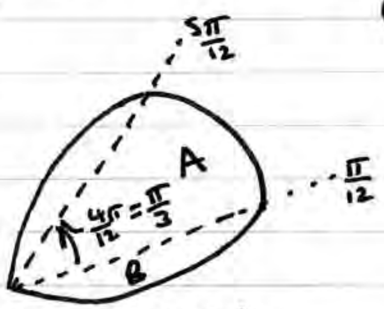
(b) Find the area of the shaded region R, giving your answer in the form $\frac{1}{12}a^2\left(p\pi + q\sqrt{3}\right)$, where p and q are integers to be found.

(7)

a) $\cancel{a} = 2\cancel{a}\sin 2\theta \quad \therefore \sin 2\theta = \frac{1}{2} \quad \therefore 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$(a, \frac{\pi}{12})\,;\,(a, \frac{5\pi}{12})$.

b)



Area = A+2B

$A = \frac{1}{2}(a)^2\left(\frac{\pi}{3}\right) = \frac{\pi}{6}a^2$

$B = \frac{1}{2}\int_0^{\frac{\pi}{12}} (2a\sin 2\theta)^2\,d\theta = 2a^2\int_0^{\frac{\pi}{12}} \sin^2 2\theta\,d\theta$

$B = 2a^2\int \frac{1}{2} - \frac{1}{2}\cos 4\theta\,d\theta = a^2\int_0^{\frac{\pi}{12}} 1 - \cos 4\theta\,d\theta$

$= a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{12}} = a^2\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$

$\therefore R = a^2\left(\frac{\pi}{6} + 2\left(\frac{\pi}{12}\right) - 2\left(\frac{\sqrt{3}}{8}\right)\right) = a^2\left(\frac{\pi}{3} + - \frac{\sqrt{3}}{4}\right)$

$R = \frac{1}{12}a^2(4\pi - 3\sqrt{3})$

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